Hence $\mathrm{mod}(a_n^2, 5) \in \{1, 4\}$ for all $n \in \mathbb{N}$.
Let's take a look on $\mathrm{mod}(L_n, 5)$, Here's a lemma :

$$\mathrm{mod}(L_n, 5) = \begin{cases} 2 & \text{if } \mathrm{mod}(n, 4) = 0; \\ 1 & \text{if } \mathrm{mod}(n, 4) = 1; \\ 3 & \text{if } \mathrm{mod}(n, 4) = 2; \\ 4 & \text{if } \mathrm{mod}(n, 4) = 3. \end{cases} \qquad \text{(lemma 1)}$$

Proof : for $n \in \{0, 1, 2, 3\}$ it is clearly true. suppose it is true for some integers $4k, 4k+1, 4k+2, 4+3$, then

$$\begin{aligned} \mathrm{mod}(L_{4k+4}, 5) &= \mathrm{mod}(L_{4k+3} + L_{4k+2}, 5) = \mathrm{mod}(4+3, 5) = 2 \\ \mathrm{mod}(L_{4k+5}, 5) &= \mathrm{mod}(L_{4k+4} + L_{4k+3}, 5) = \mathrm{mod}(2+4, 5) = 1 \\ \mathrm{mod}(L_{4k+6}, 5) &= \mathrm{mod}(L_{4k+5} + L_{4k+4}, 5) = \mathrm{mod}(1+2, 5) = 3 \\ \mathrm{mod}(L_{4k+7}, 5) &= \mathrm{mod}(L_{4k+6} + L_{4k+5}, 5) = \mathrm{mod}(3+1, 5) = 4 \end{aligned}$$

Which means that (lemma 1) is true for any integer n by strong induction.
For $n = 0$, we have $\mathrm{mod}(b_0, 5) \in \{2, 3\}$, which means that $b_0 \notin \{0, 1, 4\}$, then b_0 is not a perfect square.
For $n > 1$, we have $2^n = 4k$ where $k \in \mathbb{N}_{>0}$, therefore : $\mathrm{mod}(L_{2^n}, 5) = 2$, which implies that

$$\mathrm{mod}(b_n, 5) = \mathrm{mod}(2a_n^2, 5) \in \{2, 3\}$$

Which means that $b_n \notin \{0, 1, 4\}$, then b_n is not a perfect square for $n > 1$.
Conclusion : b_n is not a perfect square for any $n \in \mathbb{N}\backslash\{1\}$.

Also solved by Omran Kouba, Higher Institute for Applied sciences and Technology, Damascus, Syria; and the proposer.

93. *Proposed by Anastasios Kotronis, Athens, Greece.* For $x \in (-1, 1)$, evaluate

$$\sum_{n=1}^{+\infty} (-1)^{n+1} n \left(\tan^{-1} x - x + \frac{x^3}{3} - \cdots + (-1)^{n+1} \frac{x^{2n+1}}{2n+1} \right).$$

Solution 1 by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.
Let

$$f_n(x) = \tan^{-1} x - x + \frac{x^3}{3} - \cdots + (-1)^{n+1} \frac{x^{2n+1}}{2n+1}$$

Clearly we have

$$f_n'(x) = \frac{1}{1+x^2} - \sum_{k=0}^{2n} (-x^2)^k = \frac{1}{1+x^2} - \frac{1-(-x^2)^{n+1}}{1+x^2} = \frac{(-1)^{n+1} x^{2n+2}}{1+x^2}$$

Thus

$$f_n(x) = (-1)^{n+1} \int_0^x \frac{t^{2n+2}}{1+t^2} dt$$

It follows that

$$\sum_{n=1}^{\infty}(-1)^{n+1}nf_n(x) = \frac{1}{2}\int_0^x \frac{t^3}{1+t^2}\left(\sum_{n=1}^{\infty}(2n)t^{2n-1}\right)dt$$
$$= \frac{1}{2}\int_0^x \frac{t^3}{1+t^2}\left(\sum_{n=0}^{\infty}t^{2n}\right)' dt$$
$$= \frac{1}{2}\int_0^x \frac{t^3}{1+t^2}\left(\frac{1}{1-t^2}\right)' dt$$
$$= \int_0^x \frac{t^4}{(1+t^2)(1-t^2)^2}dt$$

Finally, noting that

$$\frac{t^4}{(1+t^2)(1-t^2)} = \frac{1}{8}\left(\frac{1}{(1+t)^2}+\frac{1}{(1-t)^2}-\frac{2}{1-t}-\frac{2}{1+t}+\frac{2}{1+t^2}\right)$$

we conclude that

$$\sum_{n=1}^{\infty}(-1)^{n+1}nf_n(x) = \frac{1}{4}\left(\frac{x}{1-x^2}+\ln\left(\frac{1-x}{1+x}\right)+\arctan x\right)$$

which is the desired conclusion.

Solution 2 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy. Clearly

$$\sum_{n=1}^{\infty}(-1)^{n+1}n\left(\tan^{-1}x - x + \frac{x^3}{3} - \ldots + (-1)^{n+1}\frac{x^{2n+1}}{2n+1}\right) =$$
$$= \sum_{n=1}^{\infty}(-1)^{n+1}n\sum_{k=n+1}^{\infty}(-1)^{k+1}\frac{x^{2k+1}}{2k+1}$$

Since $|x|<1$, we can rearrange the series as

$$\sum_{n=1}^{\infty}(-1)^{n+1}n\sum_{k=n+1}^{\infty}(-1)^{k+1}\frac{x^{2k+1}}{2k+1} = \sum_{k=2}^{\infty}(-1)^k\frac{x^{2k+1}}{2k+1}\sum_{n=1}^{k-1}(-1)^n n$$

$\sum_{n=1}^{k-1}(-1)^n n$ is equal to $-k/2$ if k is even and $(k-1)/2$ if k is odd. It follows

$$\sum_{k=2}^{\infty}(-1)^k\frac{x^{2k+1}}{2k+1}\sum_{n=1}^{k-1}(-1)^n n =$$
$$= -\sum_{k=1}^{\infty}\frac{kx^{4k+1}}{4k+1} - \frac{1}{2}\sum_{k=2}^{\infty}(2k-1)\frac{x^{4k-1}}{4k-1} + \frac{1}{2}\sum_{k=2}^{\infty}\frac{x^{4k-1}}{4k-1} =$$
$$= -\sum_{k=1}^{\infty}x^{4k+1} + \frac{1}{4}\sum_{k=1}^{\infty}\frac{x^{4k+1}}{4k+1} - \frac{1}{4}\sum_{k=2}^{\infty}x^{4k-1} + \frac{3}{4}\sum_{k=2}^{\infty}\frac{x^{4k-1}}{4k-1}.$$

$$\sum_{k=1}^{\infty} x^{4k+1} = \frac{x^5}{1-x^4}.$$

$$\sum_{k=1}^{\infty} \frac{x^{4k+1}}{4k+1} = \sum_{k=1}^{\infty} x^{4k+1} \int_0^1 y^{4k} dy =$$
$$= x \int_0^1 \frac{x^4 y^4}{1-x^4y^4} dy = \int_0^x \frac{y^4}{1-y^4} dy = -x + \frac{1}{4} \ln \frac{1+x}{1-x} + \frac{1}{2} \arctan x.$$

Moreover

$$\sum_{k=2}^{\infty} x^{4k-1} = \frac{x^7}{1-x^4}.$$

$$\sum_{k=2}^{\infty} x^{4k-1} \int_0^1 y^{4k-2} dy = \frac{1}{x} \int_0^1 \frac{x^8 y^6}{1-(xy)^4} dy = \int_0^x \frac{t^6}{1-t^4} dt =$$
$$= \int_0^x \left(\frac{1}{2} - \frac{1}{2} \frac{1}{1+t^2} + \frac{1}{4} \frac{t^2}{1+t} + \frac{1}{4} \frac{1}{1-t} - \frac{t+1}{4} - t^2 \right) dt =$$
$$= \frac{x}{2} - \frac{1}{2} \arctan x + \frac{x^2}{8} - \frac{x}{4} + \frac{1}{4} \ln(1+x) - \frac{1}{4} \ln(1-x) - \frac{x^2}{8} - \frac{x}{4} - \frac{x^3}{3} =$$
$$-\frac{1}{2} \arctan x + \frac{1}{4} \ln(1+x) - \frac{1}{4} \ln(1-x) - \frac{x^3}{3}.$$

By summing up the three contributions we obtain

$$-\frac{1}{4} \frac{x^5}{1-x^4} - \frac{x}{4} + \frac{1}{16} \ln \frac{1+x}{1-x} + \frac{1}{8} \arctan x - \frac{1}{4} \frac{x^7}{1-x^4} +$$
$$+\frac{3}{4} \left(-\frac{1}{2} \arctan x + \frac{1}{4} \ln(1+x) - \frac{1}{4} \ln(1-x) - \frac{x^3}{3} \right) =$$
$$= -\frac{1}{4} \frac{x}{1-x^2} - \frac{1}{4} \arctan x - \frac{1}{4} \ln \frac{1-x}{1+x}.$$

Solution 3 by Arkady Alt, San Jose, California, USA.

Let $S(x) := \sum\limits_{n=1}^{+\infty} (-1)^{n+1} n \left(\tan^{-1}(x) - \left(x - \frac{x^3}{3} + ... + (-1)^n \frac{x^{2n+1}}{2n+1} \right) \right)$.

Then $S'(x) = \sum\limits_{n=1}^{+\infty} (-1)^{n+1} n \left(\frac{1}{1+x^2} - \left(1 - x^2 + ... + \left(-x^2\right)^n\right) \right) =$

$\sum\limits_{n=1}^{+\infty} (-1)^{n+1} n \left(\frac{1}{1+x^2} - \frac{1-\left(-x^2\right)^{n+1}}{1+x^2} \right) = \sum\limits_{n=1}^{+\infty} n \cdot \frac{x^{2(n+1)}}{1+x^2} = \frac{x^4}{1+x^2} \sum\limits_{n=1}^{+\infty} n \left(x^2\right)^{n-1} =$

$\frac{x^4}{1+x^2} \cdot \frac{1}{\left(1-x^2\right)^2} = \frac{1}{4(x-1)} - \frac{1}{4(x+1)} + \frac{1}{8(x-1)^2} + \frac{1}{8(x+1)^2} + \frac{1}{4(x^2+1)}$

and, therefore, $S(x) = \int_0^x \frac{t^4 dt}{(1+t^2)(1-t^2)^2} = \frac{1}{4} \left(\ln \frac{1-t}{1+t} + \frac{t}{1-t^2} + \tan^{-1}(t) \right)_0^x =$

$\frac{1}{4} \left(\ln \frac{1-x}{1+x} + \frac{x}{1-x^2} + \tan^{-1}(x) \right)$.